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RESEARCH ARTICLE

Power-law scalings in weakly-interacting Bose gases at quantum criticality

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Supporting Information

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Supplemental Text
Figure S1
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1 Numerical methods and results for solving the TBA equation in 1D

1.1 Numerical methods

To facilitate numerical computations, the scaled dressed energy $\tilde{\epsilon}$ is rewritten as follows:

$$\begin{aligned}
&\tilde{\epsilon}(\tilde{q},\tilde{\mu},\tilde{c}) \\
&\equiv \tilde{q}^2-\tilde{\mu}-\int_{-\infty}^{\infty}\frac{d\tilde{l}}{2\pi}\frac{2\tilde{c}}{\tilde{c}^2+(\tilde{q}-\tilde{l})^2}\ln\left[1+e^{-\tilde{\epsilon}(\tilde{l},\tilde{\mu},\tilde{c})}\right] \\
&= \tilde{q}^2-\tilde{\mu}-\int_{0}^{\infty}\frac{d\tilde{l}}{2\pi}W(\tilde{l},\tilde{q},\tilde{c})\ln\left[1+e^{-\tilde{\epsilon}(\tilde{l},\tilde{\mu},\tilde{c})}\right] \quad \text{(S1)}
\end{aligned}$$

where $\tilde{q}$ and $\tilde{l}$ are scaled quasi-momenta, and the function W is defined as

$$W(\tilde{l},\tilde{q},\tilde{c}) = \frac{2\tilde{c}}{\tilde{c}^2+(\tilde{l}-\tilde{q})^2}+\frac{2\tilde{c}}{\tilde{c}^2+(\tilde{l}+\tilde{q})^2} \quad \text{(S2)}$$

It has been proved that the TBA equation can be solved iteratively [2]. The key to performing efficient numerical computations here is to identify the major contribution that is difficult to derive analytically for the integral in Eq. S1.

As $Q(\tilde{q},\tilde{l},\tilde{\mu},\tilde{c}) = W(\tilde{q},\tilde{l},\tilde{c})\ln\left[1+e^{-\tilde{\epsilon}(\tilde{q},\tilde{\mu},\tilde{c})}\right]$ decreases exponentially for large $\tilde{q}$ values, an integration over a finite range of $\tilde{q}$ already dominates the full integration to a high precision. Practically, we verified that an integral of Q over $5 < \tilde{q} < \infty$ is on the order of 10^{-10} or lower, which is negligible in typical computations. Therefore, we choose $[0,\tilde{q}_{\max}] = [0,5]$ as the integral region.

For a single iteration, instead of doing the integration in the TBA equation, we compute the Riemann sum of the function $Q(\tilde{q},\tilde{l},\tilde{\mu},\tilde{c})$ with respect to an isometric tagged partition of the scaled quasi-momentum:

$$\begin{aligned}
\tilde{q}_m &= \Delta\tilde{q}\times m, \quad m\in\mathbb{Z} \\
S\left[Q(\tilde{q},\tilde{l},\tilde{\mu},\tilde{c})\right] &= \sum_{m=0}^{\tilde{q}_{\max}/\Delta\tilde{q}} Q(\tilde{q}_m,\tilde{l},\tilde{\mu},\tilde{c})(\tilde{q}_{m+1}-\tilde{q}_m). \quad \text{(S3)}
\end{aligned}$$

Due to the Riemann-integrability of Q, the discrete Riemann sum S approaches the corresponding integral as the scaled momentum spacing $\Delta\tilde{q}$ reduces to zero. We perform computations under several $\Delta\tilde{q}$ values, and use an extrapolation to $\Delta\tilde{q} = 0$ in order to remove the $\Delta\tilde{q}$ dependence of the results.

The above method works very well for $\tilde{c} \geq 10^{-2}$. However, in order to reach a certain computation precision, the required computing time increases significantly as $\tilde{c}$ approaches zero for two reasons. First, the needed $\Delta\tilde{q}$ to reach the precision goal becomes smaller as $\tilde{c}$ decreases, resulting a larger number $(\tilde{q}_{\max}/\Delta\tilde{q})$ of intervals. Second, when using an iterative method to solve the TBA equation under a decreasing $\tilde{c}$, the change of the scaled dressed energy function after one iteration decreases correspondingly. Thus, in order to reach a certain precision, the number of iterations inevitably increases a lot. With increased computation time, the cases for $10^{-3} \leq \tilde{c} < 10^{-2}$ can also be numerically solved. Without significant simplifications of the computing algorithm, it is rather difficult to compute the scaled dressed energy for $\tilde{c}$ below 10^{-3}.

To simplify the computing tasks at small $\tilde{c}$, we first note that exactly at $\tilde{c} = 0$, the dressed energy takes the non-interacting form [2]:

$$\begin{aligned}
\tilde{\epsilon}(\tilde{q},\tilde{\mu},\tilde{c}=0) &\equiv \tilde{\epsilon}_0(\tilde{q},\tilde{\mu}) \\
&= \ln\left[e^{\tilde{q}^2-\tilde{\mu}}-1\right]. \quad \text{(S4)}
\end{aligned}$$

At a certain small $\tilde{c}$, one expect that the large-scaled-quasi-momentum behavior of $\tilde{\epsilon}$ is dominated by that of $\tilde{\epsilon}_0$. We thus introduce a characteristic scaled quasi-momentum $\tilde{q}_{\text{NI}}$ as a function of $\tilde{c}$, such that at the quantum critical point $\tilde{\mu} = 0$, the dressed energy takes the non-interacting form for those scaled quasi-momenta that satisfy $|\tilde{q}| \geq \tilde{q}_{\text{NI}}$:

$$\begin{aligned}
\tilde{\epsilon}(\tilde{q},\tilde{\mu}=0,\tilde{c}) &= \tilde{\epsilon}_0(\tilde{q},\tilde{\mu}=0) \\
&= \ln\left[e^{\tilde{q}^2}-1\right], \quad |\tilde{q}| \geq \tilde{q}_{\text{NI}} \quad \text{(S5)}
\end{aligned}$$

Based on Eq. S5, we greatly reduce the computation intensity of solving the TBA equation, and thus numerically solve this equation to obtain the scaled dressed energy at the remaining small scaled momenta in the range of $|\tilde{q}| \le \tilde{q}_{\rm NI}$.

The assumption of Eq. S5 can be numerically verified by comparing a full computation and a computation using Eq. S5. We indeed observe that proper values of $\tilde{q}_{\rm NI}$ can be chosen such that the computed results for typical thermodynamic quantities such as S_c/N remain accurate to six significant digits (relative error $\sim 10^{-6}$ or lower). For example, for $\tilde{c} = 0.005, 0.002, 0.001, 0.0005$, it is sufficient to choose $\tilde{q}_{\rm NI} = 2.5, 1.75, 1.6, 1.4$, respectively. As the interaction strength decreases to a smaller value, we start the computation by using the $\tilde{q}_{\rm NI}$ value determined to be "safe" for the previous larger $\tilde{c}$, and further perform computations under several decreasing $\tilde{q}_{\rm NI}$ values. In this way, we determine a new "safe" $\tilde{q}_{\rm NI}$ value for the current $\tilde{c}$. As $\tilde{c}$ reduces towards zero, the corresponding $\tilde{q}_{\rm NI}$ also decreases towards zero. We also verify that the choice of $\tilde{q}_{\rm NI}$ is fairly independent of the choice of scaled momentum spacing $\Delta\tilde{q}$.

Below we further explain the process of numerically solving the TBA equation under Eq. S5. For $|\tilde{q}| \le \tilde{q}_{\rm NI}$, we rewrite the TBA equation as

$$\begin{aligned}
&\tilde{\epsilon}(\tilde{q}, \tilde{\mu} = 0, \tilde{c}) \\
&= \tilde{q}^2 - \frac{1}{2\pi}\int_0^\infty {\rm d}\tilde{l}\, W(\tilde{q}, \tilde{l}, \tilde{c}) \ln(1 + e^{-\tilde{\epsilon}(\tilde{l}, 0, \tilde{c})}) \\
&= \tilde{q}^2 - \frac{1}{2\pi}\int_0^{\tilde{q}_{\rm NI}} {\rm d}\tilde{l}\, W(\tilde{q}, \tilde{l}, \tilde{c}) \ln(1 + e^{-\tilde{\epsilon}(\tilde{q}, 0, \tilde{c})}) \\
&\quad - \frac{1}{2\pi}\int_{\tilde{q}_{\rm NI}}^\infty {\rm d}\tilde{l}\, W(\tilde{q}, \tilde{l}, \tilde{c}) \ln\left(1 + \frac{1}{e^{\tilde{l}^2} - 1}\right) \\
&\equiv \tilde{q}^2 - I_1(\tilde{q}, \tilde{q}_{\rm NI}, \tilde{c}) - I_2(\tilde{q}, \tilde{q}_{\rm NI}, \tilde{c}). \qquad \text{(S6)}
\end{aligned}$$

Here, I_2 can be conveniently computed by performing a numerical integration for each $\tilde{q}$ at a given $\tilde{c}$ and a properly chosen $\tilde{q}_{\rm NI}$. It is I_1 that needs to be computed via a large number of iterations.

To further improve the numerical integrations to an accuracy better than Eq. S3, we introduce a linear interpolation for the function $\ln(1 + e^{-\tilde{\epsilon}(\tilde{q}, \tilde{\mu}, \tilde{c})})$ and analytically derive an approximate form of the integral as follows:

$$\begin{aligned}
&S_1[Q(\tilde{q}, \tilde{l}, \tilde{\mu}, \tilde{c})] \\
&= \sum_{m=0}^{\tilde{q}_{\max}/\Delta\tilde{q}} \int_{\tilde{q}_m}^{\tilde{q}_{m+1}} {\rm d}\tilde{q}\, W(\tilde{q}, \tilde{l}, \tilde{c}) \ln(1 + e^{-\tilde{\epsilon}(\tilde{q}, \tilde{\mu}, \tilde{c})}) \\
&\approx \sum_{m=0}^{\tilde{q}_{\max}/\Delta\tilde{q}} \int_{\tilde{q}_m}^{\tilde{q}_{m+1}} {\rm d}\tilde{q}\, W(\tilde{q}, \tilde{l}, \tilde{c}) \\
&\quad \times \frac{X_m(\tilde{q}_{m+1} - \tilde{q}) + X_{m+1}(\tilde{q} - \tilde{q}_m)}{\Delta\tilde{q}} \\
&\equiv \sum_{m=0}^{\tilde{q}_{\max}/\Delta\tilde{q}} I_m \qquad \text{(S7)}
\end{aligned}$$

where $X_m \equiv \ln(1 + e^{-\tilde{\epsilon}(\tilde{q}_m, \tilde{\mu}, \tilde{c})})$, and the integral I_m is given by

$$\begin{aligned}
I_m &= 2(a_m + b_m\tilde{l})\left[\arctan\left(\frac{\tilde{q}_{m+1} - \tilde{l}}{\tilde{c}}\right) - \arctan\left(\frac{\tilde{q}_m - \tilde{l}}{\tilde{c}}\right)\right] \\
&+ 2(a_m - b_m\tilde{l})\left[\arctan\left(\frac{\tilde{q}_{m+1} + \tilde{l}}{\tilde{c}}\right) - \arctan\left(\frac{\tilde{q}_m + \tilde{l}}{\tilde{c}}\right)\right] \\
&+ b_m\tilde{c}\left[\ln\frac{(\tilde{q}_{m+1} - \tilde{l})^2 + \tilde{c}^2}{(\tilde{q}_m - \tilde{l})^2 + \tilde{c}^2} + \ln\frac{(\tilde{q}_{m+1} + \tilde{l})^2 + \tilde{c}^2}{(\tilde{q}_m + \tilde{l})^2 + \tilde{c}^2}\right], \qquad \text{(S8)}
\end{aligned}$$

with the following coefficients a_m and b_m:

$$a_m = \frac{X_m\tilde{q}_{m+1} - X_{m+1}\tilde{q}_m}{\Delta\tilde{q}}, \qquad \text{(S9)}$$

$$b_m = \frac{X_{m+1} - X_m}{\Delta\tilde{q}}. \qquad \text{(S10)}$$

By introducing a characteristic $\tilde{q}_{\rm NI}$ and the above partial interpolation method, we effectively reduce the computing time for reaching a certain precision, which enables us to determine the scaled dressed energy, the critical entropy per particle and other thermodynamic quantities for very small $\tilde{c}$ (down to 5×10^{-8}).

1.2 The critical scaled compressibility

Based on the algorithm described in the previous subsection, we numerically solve the TBA equation and extract thermodynamic quantities. In addition to S_c/N and $\tilde{n}_c$, the scaled critical compressibility $\tilde{\kappa}_c \equiv \left.\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mu}}\right|_{\tilde{\mu}=0}$ is computed and shown in Figure S1. We observe a power-law scaling of $\tilde{\kappa}_c$ with respect to $\tilde{c}$, namely $\tilde{\kappa}_c \propto \tilde{c}^{\beta_{\tilde{\kappa}}}$, with a scaling exponent $\beta_{\tilde{\kappa}} = -0.993 \pm 0.005$, which agrees well with the prediction of Eq. 16 and provides clear evidence for our generic approach of studying these weakly interacting Bose gases at the quantum critical point.

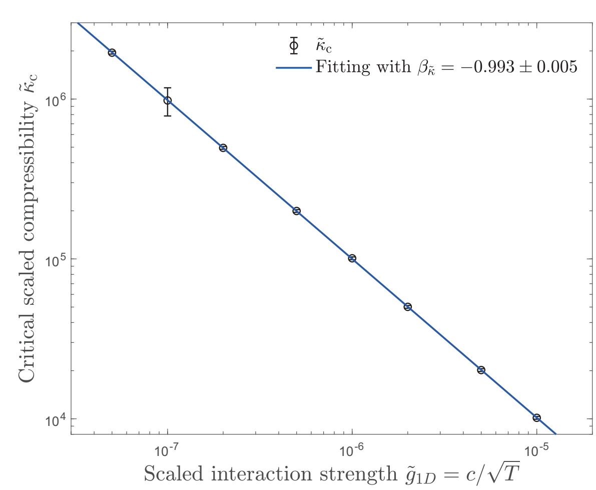

Figure S1 Critical scaled compressibility $\tilde{\kappa}_c$ of a 1D interacting Bose gas as a function of scaled interaction strength $\tilde{g}_{1D} = \tilde{c}$.

2 An analytical derivation for the $\beta_{1D} = 1/3$ power-law scaling under an infrared momentum cut-off approximation)

In this section we analytically derive the $\beta_{1D} = 1/3$ power-law scaling in the small-interaction limit by solving the TBA equation under a one-parameter infrared momentum cut-off approximation.

We first notice that for $\tilde{c} \to 0$ and any finite $\tilde{q}$, the scaled dressed energy $\tilde{\epsilon}(\tilde{q}, \tilde{\mu}, \tilde{c})$ approaches its non-interacting limit, $\tilde{\epsilon}_0(\tilde{q}, \tilde{\mu}) = \ln(e^{\tilde{q}^2-\tilde{\mu}} - 1)$. Deviation from the non-interacting form happens only for infinitesimal $\tilde{q}$ values. We thus aim to determine the infinitesimal-$\tilde{q}$ behavior of the scaled dressed energy at the critical point $\tilde{\mu} = 0$.

We make an approximation to $\tilde{\epsilon}(\tilde{q}, \tilde{\mu} = 0, \tilde{c})$ by introducing a characteristic scaled momentum $\tilde{q}_*(\tilde{c})$ that serves as a "flat-bottom" infrared momentum cut-off such that

$$\lim_{\tilde{c}\to 0} \tilde{q}_* = 0 \tag{S11}$$

and

$$\tilde{\epsilon}_{\mathrm{FB}}(\tilde{q}, \tilde{\mu} = 0, \tilde{c}) = \begin{cases} \ln\left(e^{\tilde{q}_*^2} - 1\right), & |\tilde{q}| \le \tilde{q}_* \\ \ln\left(e^{\tilde{q}^2} - 1\right), & |\tilde{q}| \ge \tilde{q}_* \end{cases} \tag{S12}$$

For simplicity, we ignore the subscript "FB" (denoting "flat bottom") in the following parts of this section. Because $\tilde{q}_*$ is the only parameter in this approximation, we solve it by writing the TBA equation at $\tilde{q} = 0$:

$$\begin{aligned} &\ln\left(e^{\tilde{q}_*^2} - 1\right) \\ &= \tilde{\epsilon}(\tilde{q} = 0, \tilde{\mu} = 0, \tilde{c}) \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tilde{l}\frac{2\tilde{c}}{\tilde{c}^2 + \tilde{l}^2}\ln\left(1 + e^{-\tilde{\epsilon}(\tilde{l}, \tilde{\mu}=0, \tilde{c})}\right) \end{aligned} \tag{S13}$$

Based on Eq. S13, we obtain that

$$\tilde{q}_*^2 = \frac{4}{\pi}\int_{\tilde{q}_*}^{\infty} \mathrm{d}\tilde{l}\left(\arctan\frac{\tilde{c}}{\tilde{l}}\right)\frac{\tilde{l}}{e^{\tilde{l}^2} - 1} \tag{S14}$$

We note that part of the integrand, $\frac{\tilde{l}}{e^{\tilde{l}^2}-1}$, behaves roughly as $1/\tilde{l}$ for small $\tilde{l}$ values on the order of $\tilde{q}_*$. This suggests that, unless $\arctan\frac{\tilde{c}}{\tilde{l}}$ provides a sufficiently small suppression factor, the integral on the right side of Eq. S14 will diverge as $\ln(\tilde{q}_*)$, which cannot match the vanishing $\tilde{q}_*^2$ on the left side of Eq. S14. Therefore, we conclude that $\tilde{c}$ must be a higher-order infinitesimal with respect to $\tilde{q}_*$, namely,

$$\lim_{\tilde{c}\to 0}\frac{\tilde{c}}{\tilde{q}_*} = 0. \tag{S15}$$

Thus for $|\tilde{l}| \ge \tilde{q}_*$, the function $\arctan(\tilde{c}/\tilde{l})$ can be approximated by $\tilde{c}/\tilde{l}$, and Eq. S14 reduces to

$$\begin{aligned} \tilde{q}_*^2 &\approx \frac{4\tilde{c}}{\pi}\int_{\tilde{q}_*}^{\infty} \mathrm{d}\tilde{l}\left[\frac{1}{\tilde{l}^2} + \left(\frac{1}{e^{\tilde{l}^2} - 1} - \frac{1}{\tilde{l}^2}\right)\right] \\ &= \frac{4\tilde{c}}{\pi}\left[\frac{1}{\tilde{q}_*} + I_1\right] \\ &\approx \frac{4\tilde{c}}{\pi\tilde{q}_*}, \end{aligned} \tag{S16}$$

where $I_1 = \int_0^\infty \mathrm{d}x(-\frac{1}{x^2} + \frac{1}{e^{x^2}-1}) \approx -1.294$. From the above equation, we derive the characteristic scaling relation of $\tilde{q}_*$ with respect to $\tilde{c}$:

$$\tilde{q}_* \approx \left(\frac{4}{\pi}\right)^{1/3}\tilde{c}^{1/3} \tag{S17}$$

Accordingly, we obtain the scaled dressed energy at zero quasi-momentum and at the quantum critical point:

$$\tilde{\epsilon}(\tilde{q} = 0, \tilde{\mu} = 0, \tilde{c}) = \ln(e^{\tilde{q}_*^2} - 1) \approx \frac{2}{3}\ln(\tilde{c}) \tag{S18}$$

We further obtain the critical scaled pressure

$$\begin{aligned} \tilde{p}_{\mathrm{c}} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(1 + e^{-\tilde{\epsilon}(\tilde{l}, 0, \tilde{c})})\mathrm{d}\tilde{l} \\ &= I_0 - \frac{2}{\pi}\int_0^{\tilde{q}_*}\frac{\tilde{l}^2}{e^{\tilde{l}^2} - 1}\mathrm{d}\tilde{l} \\ &\approx I_0 - \left(\frac{32}{\pi^4}\right)^{1/3}\tilde{c}^{1/3}, \end{aligned} \tag{S19}$$

where $I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(1 + 1/(e^{x^2} - 1))\mathrm{d}x \approx 0.736937$.

We then compute the critical scaled density as follows:

$$\tilde{n}_{\mathrm{c}} \equiv \left[\frac{\partial\tilde{p}}{\partial\tilde{\mu}}\right]_{\tilde{\mu}=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\tilde{\epsilon}}}{1 + e^{-\tilde{\epsilon}}}(-\frac{\partial\tilde{\epsilon}}{\partial\tilde{\mu}})\mathrm{d}\tilde{l}. \tag{S20}$$

To compute $\frac{\partial\tilde{\epsilon}}{\partial\tilde{\mu}}$, we perform differentiation to both sides of the TBA equation, and obtain

$$\frac{\partial\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}{\partial\tilde{\mu}} \approx -1 - \frac{-e^{-\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}}{1 + e^{-\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}}\frac{\partial\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}{\partial\tilde{\mu}} \tag{S21}$$

Thus

$$\frac{\partial\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}{\partial\tilde{\mu}} \approx -1 - e^{-\tilde{\epsilon}(\tilde{l}, \tilde{\mu}, \tilde{c})}, \tag{S22}$$

and we obtain

$$\begin{aligned} \tilde{n}_{\mathrm{c}} &\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\tilde{\epsilon}(\tilde{l}, \tilde{\mu}=0, \tilde{c})}\mathrm{d}\tilde{l} \\ &\approx \frac{1}{\pi}\left(\frac{2}{\tilde{q}_*} + I_1\right) \\ &\approx \left(\frac{2}{\pi^2}\right)^{1/3}\tilde{c}^{-1/3} \end{aligned} \tag{S23}$$

Based on the above results, the critical entropy per particle is given by

$$\frac{S_{\rm c}}{N} = \frac{3}{2}\frac{\tilde{p}_{\rm c}}{\frac{\partial\tilde{p}}{\partial\tilde{\mu}}(\tilde{\mu}=0,\tilde{c})} - \frac{\tilde{c}}{2}\frac{\partial\tilde{p}}{\partial\tilde{c}}\Big/\frac{\partial\tilde{p}}{\partial\tilde{\mu}} \approx \frac{3I_0}{2}\left(\frac{\pi^2}{2}\right)^{1/3}\tilde{c}^{1/3}, \quad (S24)$$

where the second term is much smaller than the first term for small $\tilde{c}$ and is thus neglected.

Other thermodynamic observables can be similarly computed based on their relation to the pressure and Eq. S19. For example, the critical scaled compressibility can be derived to be

$$\tilde{\kappa}_{\rm c} = \left.\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mu}}\right|_{\tilde{\mu}=0} \approx \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-\tilde{\epsilon}(\tilde{l},\tilde{\mu}=0,\tilde{c})} + e^{-2\tilde{\epsilon}(\tilde{l},\tilde{\mu}=0,\tilde{c})}\right]{\rm d}\tilde{l} \approx \frac{4}{3\pi}\frac{1}{\tilde{q}_*^3} \propto \frac{1}{\tilde{c}} = \frac{1}{\tilde{g}_{\rm 1D}}, \quad (S25)$$

where lower-order constant terms and those proportional to $1/\tilde{q}_*$ are neglected.

Eqs. S24 and S25 show that in the small interaction strength limit and under this one-parameter infrared momentum cut-off approximation, the critical entropy per particle $S_{\rm c}/N$ and critical scaled compressibility $\tilde{\kappa}_{\rm c}$ both obey simple power-law scalings with respect to the scaled interaction strength, with scaling exponents being $1/3$ and -1, respectively. These analytical results serve as independent and complementary evidences for the generic predictions in the main text (Eqs. 13, 14 and 16) for interacting Bose gases at a quantum critical point.

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