Appendices for:

Theoretical study of failure in composite pressure vessels subjected to low-velocity impact and internal pressure

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Appendix A. Coefficients of Eq. (3)

$$\omega_{1,2}^2=\frac{1}{2}\left(\frac{K_1+K_2}{M_1}+\frac{K_2}{M_2}\right)\mp\sqrt{-\frac{1}{4}\left(\frac{K_1+K_2}{M_1}-\frac{K_2}{M_2}\right)^2+\frac{K_2^2}{M_1M_2}}$$
$$C_1=\frac{K_2}{K_2-\omega_1^2M_2}$$
$$C_2=\frac{K_2}{K_2-\omega_2^2M_2} \qquad \text{(A1)}$$
$$A_1=\frac{V}{\omega_1\left(C_2-C_1\right)}$$
$$A_2=\frac{V}{\omega_2\left(C_1-C_2\right)}$$

where K_2 is the effective stiffness [35], K_1 the equivalent stiffness of the simply-supported laminated shell, M_2 is impactor mass and M_1 is the effective mass of laminated shell [35]. V represents the velocity of impactor.

Appendix B. Virtual energy terms

The virtual energies reflected in Eq. (10) are expressed in terms of displacement field as below:

$$\delta K=\int_{\Omega}\int_{-h/2}^{h/2}\rho_0(\dot{u}\delta\dot{u}+\dot{v}\delta\dot{v}+\dot{w}\delta\dot{w})Rdzdxd\theta$$
$$=\int_{\Omega}\left[\sum_{I,J=1}^{N}I^{IJ}\left(\dot{U}_I\delta\dot{U}_J+\dot{V}_I\delta\dot{V}_J+\dot{W}_I\delta\dot{W}_J\right)\right]Rdxd\theta \qquad \text{(B1)}$$

$$\delta V=\int_{\Omega}\left[q(x,\theta)\delta w\left(x,y,\frac{h}{2}\right)\right]Rdxd\theta$$
$$+\int_{\Gamma}\int_{-h/2}^{h/2}\left[\hat{\sigma}_{nn}\delta u_n+\hat{\sigma}_{ns}\delta u_s+\hat{\sigma}_{nz}\delta w\right]dzds \qquad \text{(B2)}$$
$$=\int_{\Omega}\left(q_b\delta w_0\right)Rdxd\theta+\int_{\Gamma}\left[\sum_{I=1}^{N}\left(\hat{N}_{nn}^I\delta U_I^n+\hat{N}_{ns}^I\delta U_I^s+\hat{Q}_n^I\delta W_I\right)\right]ds$$

$$\delta U = \int_{\Omega}[\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_{x\theta}\delta\varepsilon_{x\theta} + \sigma_{xz}\delta\varepsilon_{xz}$$
$$+\sigma_{\theta z}\delta\varepsilon_{\theta z})dz]Rdxd\theta$$
$$= \int_{\Omega}^{1}\left\{\sum_{I,J=1}^{N}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xx}\left[\left(\frac{\partial\delta U_I}{\partial x}\right)\Phi^I + \frac{1}{2}\left(\frac{\partial\delta W_I}{\partial x}\right)\left(\frac{\partial W_J}{\partial x}\right)\Phi^I\Phi^J\right]\right.\right.$$
$$+\sigma_{\theta\theta}\left[\left(\frac{\partial\delta V_I}{\partial\theta}\right)\Phi^I + \left(\frac{\delta W_I}{R}\right)\Phi^I + \frac{1}{2}\left(\frac{\partial\delta W_I}{\partial\theta}\right)\left(\frac{\partial W_J}{\partial\theta}\right)\Phi^I\Phi^J\right]$$
$$+\sigma_{zz}\left[\delta W_I\frac{d\Phi^I}{dz}\right] + \sigma_{x\theta}\left[\left(\frac{\partial\delta U_I}{\partial\theta} + \frac{\partial\delta U_I}{\partial x}\right)\Phi^I + \left(\frac{\partial\delta w_I}{\partial x}\right)\left(\frac{\partial W_J}{\partial\theta}\right)\Phi^I\Phi^J\right] \quad \text{(B3)}$$
$$+\sigma_{xz}\left[\delta V_I\frac{d\Phi^I}{dz} + \left(\frac{\partial\delta W_I}{\partial\theta} - \frac{\delta V_I}{R}\right)\Phi^I\right]$$
$$\left.\left.+\sigma_{\theta z}\left[\delta U_I\frac{d\Phi^I}{dz} + \frac{\partial\delta W_I}{\partial x}\Phi^I\right]\right]\right\}Rdxd\theta$$

Resultant stress components are described using below equations

$$\begin{Bmatrix} N_{xx}^I \\ N_{\theta\theta}^I \\ N_{x\theta}^I \end{Bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{Bmatrix}\Phi^I dz, \quad \begin{Bmatrix} N_{xx}^{IJ} \\ N_{\theta\theta}^{IJ} \\ N_{x\theta}^{IJ} \end{Bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{Bmatrix}\Phi^I\Phi^J dz \quad \text{(B4)}$$

$$\begin{Bmatrix} Q_x^I \\ Q_\theta^I \\ Q_z^I \end{Bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_{xz} \\ \sigma_{\theta z} \\ \sigma_{zz} \end{Bmatrix}\Phi^i dz, \quad \begin{Bmatrix} \bar{Q}_x^I \\ \bar{Q}_\theta^I \\ \bar{Q}_z^I \end{Bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix} \sigma_{xz} \\ \sigma_{\theta z} \\ \sigma_{zz} \end{Bmatrix}\left(\frac{d\Phi^I}{dz}\right)dz \quad \text{(B5)}$$

Substituting (B4) and (B5) into (B3), we will obtain:

$$\delta U = \int_{\Omega}[N_{xx}^I\left(\frac{\partial\delta U_I}{\partial x}\right) + \frac{1}{2}N_{xx}^{IJ}\left(\frac{\partial W_I}{\partial x}\right)\left(\frac{\partial\delta W_J}{\partial x}\right) + N_{\theta\theta}^I\left(\frac{\partial\delta V^I}{\partial\theta} + \frac{\delta W_I}{R}\right)$$
$$+\frac{1}{2}N_{\theta\theta}^{IJ}\left(\frac{\partial W_I}{\partial\theta}\right)\left(\frac{\partial\delta W_J}{\partial\theta}\right) + \bar{Q}_z^I\left(\delta W_I\right) + N_{x\theta}^I\left(\frac{\partial\delta U_I}{\partial\theta} + \frac{\partial\delta U_I}{\partial x}\right) \quad \text{(B6)}$$
$$+N_{x\theta}^{IJ}\left(\frac{\partial W_I}{\partial\theta}\right)\left(\frac{\partial\delta W_I}{\partial x}\right) + \bar{Q}_x^I\left(\delta V_I\right) + Q_x^I\left(\frac{\partial\delta W_I}{\partial\theta} - \frac{\delta V_I}{R}\right)$$
$$+\bar{Q}_\theta^I\left(\delta U_I\right) + \bar{Q}_\theta^I\left(\frac{\partial\delta W_I}{\partial x}\right)]Rdxd\theta$$

(B1), (B2) and (B6) are substituted into Eq. (10) and performing part-by-part integral, we will obtain:

$$
\begin{aligned}
&-\int_0^T \int_\Omega I^{IJ}\left[\ddot{U}_I \delta U_J + \ddot{V}_I \delta V_J + \ddot{W}_I \delta W_J\right] R dx d\theta dt \\
&\quad + \int_\Omega I^{IJ}\left[\dot{U}_I \delta U_J + \dot{V}_I \delta V_J + \dot{W}_I \delta W_J\right] R dx d\theta \\
&\quad + \int_0^T \int_\Omega [\frac{\partial N_{xx}^I}{\partial x}\delta U_I + \frac{1}{2}\frac{\partial}{\partial x}\left[N_{xx}^{IJ} \cdot \frac{\partial w_J}{\partial x}\right]\delta W_I + \frac{\partial N_{\theta\theta}^I}{\partial \theta}\delta V_I - \frac{1}{R} N_{\theta\theta}^I \delta W \\
&\quad + \frac{1}{2}\frac{\partial}{\partial \theta}\left[N_{\theta\theta}^{IJ}\frac{\partial w_I}{\partial \theta}\right]\delta W^I - \bar{Q}_z^I \delta W_I + \frac{\partial N_{x\theta}^I}{\partial \theta}\delta U_I + \frac{\partial N_{x\theta}^I}{\partial x}\delta U_I \\
&\quad + \frac{\partial}{\partial x}\left[N_{x\theta}^{IJ} \cdot \frac{\partial W_I}{\partial \theta}\right]\delta W_I - \bar{Q}_x^I \delta V_I + \frac{\partial Q_x^I}{\partial \theta}\delta W_I + \frac{Q_x^I}{R}\delta V_I - \bar{Q}_\theta^I \delta U_I \\
&\quad + \frac{\partial Q_\theta^I}{\partial x}\delta W_I] R dx d\theta = 0
\end{aligned} \tag{B7}
$$

Recalling from Hamilton principal, virtual displacement at t = 0 and t = T are zero. Applying this assumption to (B7), equations of motion are obtained as expressed in Eq. (11).

Appendix C. Components of stiffness matrix of a lamina in cylindrical coordinate system

$$
\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta - 4Q_{16}\cos^3\theta\sin\theta + 2\left(Q_{12} + 2Q_{66}\right)\cos^2\theta\sin^2\theta - 4Q_{26}\cos\theta\sin^3\theta \\
&\quad + Q_{22}\sin^4\theta
\end{aligned} \tag{C1}
$$

$$
\begin{aligned}
\bar{Q}_{12} &= Q_{12}\cos^4\theta + 2\left(Q_{16} - Q_{26}\right)\cos^3\theta\sin\theta + \left(Q_{11} + Q_{22} - 4Q_{66}\right)\cos^2\theta\sin^2\theta \\
&\quad + 2\left(Q_{26} - Q_{16}\right)\cos\theta\sin^3\theta + Q_{12}\sin^4\theta
\end{aligned} \tag{C2}
$$

$$
\bar{Q}_{13} = Q_{13}\cos^2\theta - 2Q_{36}\cos\theta\sin\theta + Q_{23}\sin^2\theta \tag{C3}
$$

$$
\begin{aligned}
\bar{Q}_{16} &= Q_{16}\cos^4\theta + \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos^3\theta\sin\theta + 3\left(Q_{26} - Q_{16}\right)\cos^2\theta\sin^2\theta \\
&\quad + \left(2Q_{66} + Q_{12} - Q_{22}\right)\cos\theta\sin^3\theta + 3\left(Q_{26} - Q_{16}\right)\cos^2\theta\sin^2\theta \\
&\quad + \left(2Q_{66} + Q_{12} - Q_{22}\right)\cos\theta\sin^3\theta - Q_{26}\sin^4\theta
\end{aligned} \tag{C4}
$$

$$
\begin{aligned}
\bar{Q}_{22} &= Q_{22}\cos^4\theta + 4Q_{26}\cos^3\theta\sin\theta + 2\left(Q_{12} + 2Q_{66}\right)\cos^2\theta\sin^2\theta + 4Q_{16}\cos\theta\sin^3\theta \\
&\quad + Q_{11}\sin^4\theta
\end{aligned} \tag{C5}
$$

$$
\bar{Q}_{23} = Q_{23}\cos^2\theta + 2Q_{36}\cos\theta\sin\theta + Q_{13}\sin^2\theta \tag{C6}
$$

$$
\bar{Q}_{23} = Q_{23}\cos^2\theta + 2Q_{36}\cos\theta\sin\theta + Q_{13}\sin^2\theta \tag{C7}
$$

$$
\bar{Q}_{33} = Q_{33} \tag{C8}
$$

$$
\bar{Q}_{36} = \left(Q_{13} - Q_{23}\right)\cos\theta\sin\theta + Q_{36}\left(\cos^2\theta - \sin^2\theta\right) \tag{C9}
$$

$$\bar{Q}_{66} = 2\left(Q_{16} - Q_{26}\right)\cos^3\theta\sin\theta$$
$$+\left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\cos^2\theta\sin^2\theta \quad \text{(C10)}$$
$$+2\left(Q_{26} - Q_{16}\right)\cos\theta\sin^3\theta + Q_{66}\left(\cos^4\theta + \sin^4\theta\right)$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta + 2Q_{45}\cos\theta\sin\theta \quad \text{(C11)}$$

$$\bar{Q}_{45} = Q_{45}\left(\cos^2\theta - \sin^2\theta\right) + \left(Q_{55} - Q_{44}\right)\cos\theta\sin\theta \quad \text{(C12)}$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta - 2Q_{45}\cos\theta\sin\theta \quad \text{(C13)}$$

where:

$$Q_{11} = \frac{1 - \nu_{23}\nu_{32}}{E_2 E_3 \Delta} \quad \text{(C14)}$$

$$Q_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} \quad \text{(C15)}$$

$$Q_{13} = \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} \quad \text{(C16)}$$

$$Q_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1 E_3 \Delta} \quad \text{(C17)}$$

$$Q_{23} = \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_3 \Delta} \quad \text{(C18)}$$

$$Q_{33} = \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta} \quad \text{(C19)}$$

$$Q_{66} = G_{12} \quad \text{(C20)}$$

$$Q_{44} = G_{23} \quad \text{(C21)}$$

$$Q_{55} = G_{31} \quad \text{(C22)}$$

where

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3} \quad \text{(C23)}$$

For an orthotropic material Q_{16}, Q_{26}, Q_{36} and Q_{45} are zero.

Appendix D. Constitutive equation of laminated composites in the context of LWT

$$\begin{Bmatrix} N_{xx}^{I} \\ N_{\theta\theta}^{I} \\ N_{x\theta}^{I} \end{Bmatrix} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{Bmatrix} \Phi^{I} dz$$

$$= \sum_{I=1}^{N} \begin{bmatrix} A_{11}^{IJ} & A_{12}^{IJ} & \tilde{A}_{13}^{IJ} & A_{16}^{IJ} \\ A_{12}^{IJ} & A_{22}^{IJ} & \tilde{A}_{23}^{IJ} & A_{26}^{IJ} \\ A_{16}^{IJ} & A_{26}^{IJ} & \tilde{A}_{36}^{IJ} & A_{66}^{IJ} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial U_I}{\partial x} \\ \dfrac{\partial V_I}{\partial \theta} + \dfrac{W_I}{R} \\ \dfrac{\partial U_I}{\partial \theta} + \dfrac{\partial V_J}{\partial x} \end{Bmatrix}$$

$$+ \sum_{I,J=1}^{N} \begin{bmatrix} B_{11}^{IJK} & B_{12}^{IJK} & B_{16}^{IJK} \\ B_{12}^{IJK} & B_{22}^{IJK} & B_{26}^{IJK} \\ B_{16}^{IJK} & B_{26}^{IJK} & B_{66}^{IJK} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial W_I}{\partial x}\right)\left(\dfrac{\partial W_J}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial W_I}{\partial \theta}\right)\left(\dfrac{\partial W_J}{\partial \theta}\right) \\ \dfrac{\partial W_I}{\partial x}\dfrac{\partial w_J}{\partial \theta} \end{Bmatrix} \quad \text{(D1)}$$

$$\begin{Bmatrix} N_{xx}^{IJ} \\ N_{\theta\theta}^{IJ} \\ N_{x\theta}^{IJ} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{Bmatrix} \Phi^I \Phi^J dz$$

$$= \sum_{I=1}^{N} \begin{bmatrix} B_{11}^{IJK} & B_{12}^{IJK} & \tilde{B}_{13}^{IIK} & B_{16}^{IJK} \\ B_{12}^{IJK} & B_{22}^{IJK} & \tilde{B}_{23}^{IJK} & B_{26}^{IIK} \\ B_{16}^{IJK} & B_{26}^{IJK} & \tilde{B}_{36}^{IJK} & B_{66}^{IJK} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial U_I}{\partial x} \\ \dfrac{\partial V_I}{\partial \theta} + \dfrac{W_I}{R} \\ W_I \\ \dfrac{\partial U_I}{\partial \theta} + \dfrac{\partial V_J}{\partial x} \end{Bmatrix}$$

$$+ \sum_{I,J=1}^{N} \begin{bmatrix} D_{11}^{IJKP} & D_{12}^{IJKP} & D_{16}^{IJKP} \\ D_{12}^{IJKP} & D_{22}^{IJKP} & D_{26}^{IJKP} \\ D_{16}^{IJKP} & D_{26}^{IJKP} & D_{66}^{IJKP} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial W_I}{\partial x}\right)\left(\dfrac{\partial W_J}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial W_I}{\partial \theta}\right)\left(\dfrac{\partial W_J}{\partial \theta}\right) \\ \dfrac{\partial W_I}{\partial x}\dfrac{\partial w_J}{\partial \theta} \end{Bmatrix} \quad \text{(D2)}$$

$$\begin{Bmatrix} Q_x^I \\ Q_\theta^I \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{\theta z} \end{Bmatrix} \Phi^I dz$$

$$= \sum_{I=1}^{N} \left(\begin{bmatrix} \bar{B}_{55}^{IJ} & \bar{B}_{45}^{IJ} \\ \bar{B}_{45}^{IJ} & \bar{B}_{44}^{IJ} \end{bmatrix} \begin{Bmatrix} U_I \\ \left(1 - \dfrac{1}{R}\right) V_I \end{Bmatrix} + \begin{bmatrix} \bar{D}_{55}^{IJ} & \bar{D}_{45}^{IJ} \\ \bar{D}_{45}^{IJ} & \bar{D}_{44}^{IJ} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial W_I}{\partial x} \\ \dfrac{\partial W_I}{\partial \theta} \end{Bmatrix} \right) \quad \text{(D3)}$$

$$\begin{Bmatrix} \bar{Q}_x^I \\ \bar{Q}_\theta^I \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{\theta z} \end{Bmatrix} \left(\frac{d\Phi^I}{dz} \right) dz$$

$$= \sum_{I=1}^{N} \left(\begin{bmatrix} \bar{A}_{55}^{IJ} & \bar{A}_{45}^{IJ} \\ \bar{A}_{45}^{IJ} & \bar{A}_{44}^{IJ} \end{bmatrix} \begin{Bmatrix} U_I \\ \left(1 - \dfrac{1}{R}\right) V_I \end{Bmatrix} + \begin{bmatrix} \bar{B}_{55}^{IJ} & \bar{B}_{45}^{IJ} \\ \bar{B}_{45}^{IJ} & \bar{B}_{44}^{IJ} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial W_I}{\partial x} \\ \dfrac{\partial W_I}{\partial \theta} \end{Bmatrix} \right) \quad \text{(D4)}$$

$$\bar{Q}_z^I = \sum_{k=1}^{N} \int_{z_b^k}^{z_t^k} \{ \sigma_{zz} \} \left(\frac{d\Phi^I}{dz} \right) dz$$

$$= \sum_{I=1}^{N} \left[\bar{A}_{13}^{IJ} \frac{\partial U_I}{\partial x} + \bar{A}_{23}^{IJ} \left(\frac{\partial V_I}{\partial \theta} + \frac{W_I}{R} \right) + \bar{A}_{33}^{IJ} W_I + \bar{A}_{36}^{IJ} \left(\frac{\partial U_I}{\partial \theta} + \frac{\partial V_J}{\partial x} \right) \right] \quad \text{(D5)}$$

$$Q_z^I = \sum_{k=1}^{N} \int_{z_b^k}^{z_t^k} \left\{ \sigma_{zz} \right\} \Phi^I dz$$

$$= \sum_{I=1}^{N} \left[\bar{B}_{13}^{IJK} \frac{1}{2} \left(\frac{\partial W_I}{\partial x} \right) \left(\frac{\partial W_J}{\partial x} \right) + \bar{B}_{23}^{IJK} \frac{1}{2} \left(\frac{\partial W_I}{\partial \theta} \right) \left(\frac{\partial W_J}{\partial \theta} \right) + \bar{B}_{36}^{IJK} \left(\frac{\partial W_I}{\partial x} \frac{\partial w_J}{\partial \theta} \right) \right] \quad \text{(D6)}$$

where:

$$A_{ij}^{IJ} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \Phi^J dz \quad \text{(D7)}$$

$$\bar{A}_{ij}^{IJ} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \frac{d\Phi^I}{dz} \frac{d\Phi^J}{dz} dz \quad \text{(D8)}$$

$$\tilde{A}_{ij}^{IJ} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \frac{d\Phi^J}{dz} dz \quad \text{(D9)}$$

$$\bar{B}_{ij}^{IJ} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \frac{d\Phi^I}{dz} \Phi^J dz \quad \text{(D10)}$$

$$B_{ij}^{IJk} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \Phi^J \Phi^k dz \quad \text{(D11)}$$

$$\tilde{B}_{ij}^{IJk} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \Phi^J \frac{d\Phi^k}{dz} dz \quad \text{(D12)}$$

$$\bar{D}_{ij}^{IJ} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \Phi^J dz \quad \text{(D13)}$$

$$D_{ij}^{IJkp} = \sum_{K=1}^{N} \int_{z_b^k}^{z_t^k} \bar{Q}_{ij}^k \Phi^I \Phi^J \Phi^k \Phi^p dz \quad \text{(D14)}$$

Appendix E. Differential operators reflected in final governing equations for obtaining displacement field

$$L_{11} = A_{11} \frac{\partial^2 U}{\partial x^2} + A_{16} \frac{\partial^2 U}{\partial \theta \partial x} + A_{16} \frac{\partial^2 U}{\partial x \partial \theta} + A_{66} \frac{\partial^2 U}{\partial \theta^2} + A_{16} \frac{\partial^2 U}{\partial x^2} + A_{66} \frac{\partial^2 U}{\partial \theta \partial x} - B_{45} U \quad \text{(E1)}$$

$$L_{12} = A_{12} \frac{\partial^2 V}{\partial \theta \partial x} + A_{16} \frac{\partial^2 V}{\partial x^2} + A_{26} \frac{\partial^2 V}{\partial \theta^2} + A_{66} \frac{\partial^2 V}{\partial x \partial \theta} + A_{26} \frac{\partial^2 V}{\partial \theta \partial x} + A_{66} \frac{\partial^2 V}{\partial x^2} - B_{44} V \quad \text{(E2)}$$

$$L_{13} = A_{13}\frac{\partial W}{\partial x} + \frac{1}{2}B_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial x^2}\right) + \frac{1}{2}B_{12}\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) + B_{16}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)$$
$$+ A_{36}\frac{\partial W}{\partial\theta} + \frac{1}{2}B_{16}\left(\frac{\partial^2 W}{\partial x\partial\theta}\right)\left(\frac{\partial^2 W}{\partial x\partial\theta}\right) + \frac{1}{2}B_{26}\left(\frac{\partial^2 w}{\partial\theta^2}\right)\left(\frac{\partial^2 W}{\partial\theta^2}\right) + B_{66}\left(\frac{\partial^2 W}{\partial x\partial\theta}\right)\left(\frac{\partial^2 W}{\partial\theta^2}\right) \quad \text{(E3)}$$
$$+ A_{36}\left(\frac{\partial W}{\partial x}\right) + \frac{1}{2}B_{16}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial x^2}\right) + \frac{1}{2}B_{26}\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) - D_{45}\frac{\partial W}{\partial x}n - D_{44}\frac{\partial W}{\partial\theta}n$$

$$L_{21} = A_{12}\left(\frac{\partial^2 U}{\partial x\partial\theta}\right) + A_{26}\left(\frac{\partial^2 U}{\partial\theta^2}\right) - B_{55}U + \frac{1}{R}A_{55}U \quad \text{(E4)}$$

$$L_{22} = A_{22}\left(\frac{\partial^2 V}{\partial\theta^2}\right) + A_{26}\left(\frac{\partial^2 V}{\partial x\partial\theta}\right) - B_{45}V + \frac{1}{R}A_{45}V \quad \text{(E5)}$$

$$L_{23} = \mathrm{A}_{23}\left(\frac{\partial \mathrm{W}}{\partial\theta}\right) + \frac{1}{2}\mathrm{B}_{12}\left(\frac{\partial^2 \mathrm{W}}{\partial x\partial\theta}\right)\left(\frac{\partial^2 \mathrm{W}}{\partial x\partial\theta}\right) + \frac{1}{2}\mathrm{B}_{22}\left(\frac{\partial^2 \mathrm{W}}{\partial\theta^2}\right)\left(\frac{\partial^2 \mathrm{W}}{\partial\theta^2}\right) + \mathrm{B}_{26}\left(\frac{\partial^2 \mathrm{W}}{\partial x\partial\theta}\right)\left(\frac{\partial^2 \mathrm{W}}{\partial\theta^2}\right)$$
$$-\mathrm{D}_{55}\left(\frac{\partial \mathrm{W}}{\partial x}\right) - \mathrm{D}_{45}\left(\frac{\partial \mathrm{W}}{\partial\theta}\right) + \frac{1}{\mathrm{R}}\left(\frac{\partial \mathrm{W}}{\partial x}\right) + \frac{1}{\mathrm{R}}\mathrm{B}_{45}\left(\frac{\partial \mathrm{W}}{\partial\theta}\right) \quad \text{(E6)}$$

$$L_{31} = A_{45}\left(\frac{\partial U}{\partial x}\right) + A_{55}\left(\frac{\partial U}{\partial\theta}\right) - A_{13}\left(\frac{\partial U}{\partial x}\right) - A_{36}\left(\frac{\partial U}{\partial\theta}\right) + B_{11}\left(\frac{\partial^2 U}{\partial x^2}\right) + B_{16}\left(\frac{\partial^2 U}{\partial\theta\partial x}\right)$$
$$+B_{16}\left(\frac{\partial^2 U}{\partial x^2}\right) + B_{66}\left(\frac{\partial^2 U}{\partial\theta\partial x}\right) \quad \text{(E7)}$$

$$L_{32} = A_{44}\left(\frac{\partial V}{\partial x}\right) + A_{45}\left(\frac{\partial V}{\partial\theta}\right) - A_{23}\left(\frac{\partial V}{\partial x}\right) + B_{12}\left(\frac{\partial^2 V}{\partial\theta\partial x}\right) + B_{16}\left(\frac{\partial^2 V}{\partial x^2}\right)$$
$$+B_{26}\left(\frac{\partial^2 V}{\partial\theta\partial x}\right) + B_{66}\left(\frac{\partial^2 V}{\partial x^2}\right) \quad \text{(E8)}$$

$$L_{33} = B_{45}\left(\frac{\partial^2 W}{\partial x^2}\right) + B_{44}\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) + B_{55}\left(\frac{\partial^2 W}{\partial x\partial\theta}\right) + B_{45}\left(\frac{\partial^2 W}{\partial\theta^2}\right) - A_{33}W + B_{13}\left(\frac{\partial W}{\partial x}\right)$$
$$+\frac{1}{2}D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial x^2}\right) + \frac{1}{2}D_{12}\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) + D_{16}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) \quad \text{(E9)}$$
$$+B_{36}\left(\frac{\partial W}{\partial x}\right) + \frac{1}{2}D_{13}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial x^2}\right) + \frac{1}{2}D_{23}\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right) + D_{66}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial\theta\partial x}\right)$$

Appendix F. Weight coefficients for DQM C_{ijk}

$$C^1 = \frac{\left(\frac{\pi}{2}\right) P\left(x_i\right)}{P\left(x_j\right) \sin\left[\left(\frac{x_i - x_j}{2}\right)\pi\right]}, \qquad i, j = 1, \cdots, N \quad \text{(F1)}$$

$$P(x_i) = \prod_{i=1, i \neq j}^{N} \sin\left[\left(\frac{x_i - x_j}{2}\right)\pi\right] \quad \text{(F2)}$$

$$C^2 = C^1\left[-2C^1 - \pi \operatorname{ctg}\left[\left(\frac{x_i - x_j}{2}\right)\pi\right]\right] \quad \text{(F3)}$$

where:

$$x_i = \frac{i-1}{N-1} \quad \text{(F4)}$$