# Appendices for:

# Discontinuous mechanical behaviors of existing shield tunnel with stiffness reduction at longitudinal joints

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## Appendix A. List of symbols

$A$: cross-sectional area

$D$: diameter of existing shield tunnel/pipeline

$EI$: bending stiffness of beam

$E_s$: soil elastic modulus

$e_1(x)$: virtual pressure for modeling shearing dislocation of joints

$e_2(x)$: virtual pressure for modeling bending opening of joints

$F_1$: concentrated force exerting at joints

$GA$: shear stiffness of beam

$H(x–x_j)$: Heaviside step function

$I$: second moment of inertia

$i$: settlement trough width

$k$: coefficient of subgrade reaction

$L$: overall length

$l$: length of difference nodes

$l_s$: width of each segmental ring

$M_2$: a pair of couples applied at joints

$M(x)$: bending moment of beam

$M_e(x)$: virtual bending moment induced by virtual pressures

$M_j(x)$: bending moment of longitudinal joint

$Q(x)$: shear force of beam

$Q_e(x)$: virtual shear force induced by virtual pressures

$Q_j(x)$: shear force of longitudinal joint

$q(x)$: additional pressure exerting on Timoshenko beam

$r$: radius of tunnel

$s(x)$: tunneling-induced subsurface settlement

$s_{max}$: maximum subsurface settlement

$V_l$: new tunneling-induced volume loss

$w(x)$: deflection of Timoshenko beam

$x_j$: location of longitudinal joint

$Z_0$: buried depth of new tunnel

$z_p$: buried depth of pipeline

$\alpha$: coefficient of shearing stiffness reduction

$\beta$: coefficient of bending stiffness reduction

$\Delta\theta$: relative rotation angle

$\Delta\delta$: relative dislocation

$\delta' (x–x_j)$: first-order derivative of Dirac delta function

$\delta'' (x–x_j)$: second-order derivative of Dirac delta function

$\psi$: neutral-axis angle

$\mu$: soil Poisson's ratio

## Appendix B. Detailed expressions of equations, matrices, and vectors

The finite differential forms of virtual pressures $e_1(x)$ and $e_2(x)$ are as follows.

$$e_1(x_j)=\begin{cases}\dfrac{(1-\alpha)}{l}EI(\dfrac{\mathrm{d}^3w(x_j)}{\mathrm{d}x^3}-\dfrac{k}{GA}\dfrac{\mathrm{d}w(x_j)}{\mathrm{d}x}+\dfrac{1}{GA}\dfrac{\mathrm{d}q(x_j)}{\mathrm{d}x}), & (i=j-1)\\ 0 & (i=j)\\ -\dfrac{(1-\alpha)}{l}GA(\dfrac{\mathrm{d}^3w(x_j)}{\mathrm{d}x^3}-\dfrac{k}{GA}\dfrac{\mathrm{d}w(x_j)}{\mathrm{d}x}+\dfrac{1}{GA}\dfrac{\mathrm{d}q(x_j)}{\mathrm{d}x}), & (i=j+1)\end{cases} \tag{B.1}$$

$$e_2(x_j)=\begin{cases}\dfrac{(1-\beta)}{l^2}EI(\dfrac{\mathrm{d}^2w(x_j)}{\mathrm{d}x^2}-\dfrac{k}{GA}w(x_j)+\dfrac{q(x_j)}{GA}), & (i=j-1)\\ -2\dfrac{(1-\beta)}{l^2}EI(\dfrac{\mathrm{d}^2w(x_j)}{\mathrm{d}x^2}-\dfrac{k}{GA}w(x_j)+\dfrac{q(x_j)}{GA}), & (i=j)\\ \dfrac{(1-\beta)}{l^2}EI(\dfrac{\mathrm{d}^2w(x_j)}{\mathrm{d}x^2}-\dfrac{k}{GA}w(x_j)+\dfrac{q(x_j)}{GA}). & (i=j+1)\end{cases}$$

The finite differential form of the governing differential equation is as follows.

$$\begin{cases} \dfrac{6w_i-4(w_{i+1}+w_{i-1})+(w_{i+2}+w_{i-2})}{l^4}-\dfrac{k}{GA}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}+\dfrac{k}{EI}w_i \\ =\dfrac{1}{EI}q_i-\dfrac{1}{GA}\dfrac{q_{i+1}-2q_i+q_{i-1}}{l^2}, \qquad (i\neq j-1, j, j+1) \\ \\ \dfrac{6w_i-4(w_{i+1}+w_{i-1})+(w_{i+2}+w_{i-2})}{l^4}-\dfrac{1-\alpha}{l}\dfrac{-w_{i-2}+2w_{i-1}-2w_{i+1}+w_{i+2}}{2l^3} \\ -\dfrac{k}{GA}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}-\dfrac{(1-\beta)}{l^2}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}+\dfrac{k}{GA}\dfrac{1-\alpha}{l}\dfrac{w_{i+1}-w_{i-1}}{2l} \qquad (i=j-1) \\ +\dfrac{k}{EI}w_i+\dfrac{k(1-\beta)}{GAl^2}w_i=\dfrac{1}{EI}q_i+\dfrac{(1-\beta)}{GAl^2}q_i+\dfrac{1-\alpha}{GAl}\dfrac{q_{i+1}-q_{i-1}}{2l}-\dfrac{1}{GA}\dfrac{q_{i+1}-2q_i+q_{i-1}}{l^2}, \\ \\ \dfrac{6w_i-4(w_{i+1}+w_{i-1})+(w_{i+2}+w_{i-2})}{l^4}-\dfrac{k}{GA}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}+\dfrac{2(1-\beta)}{l^2}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2} \qquad (i=j) \\ +\dfrac{k}{EI}w_i-\dfrac{2k(1-\beta)}{GAl^2}w_i=\dfrac{1}{EI}q_i-\dfrac{2(1-\beta)}{GAl^2}q_i-\dfrac{1}{GA}\dfrac{q_{i+1}-2q_i+q_{i-1}}{l^2}, \\ \\ \dfrac{6w_i-4(w_{i+1}+w_{i-1})+(w_{i+2}+w_{i-2})}{l^4}+\dfrac{1-\alpha}{l}\dfrac{-w_{i-2}+2w_{i-1}-2w_{i+1}+w_{i+2}}{2l^3} \\ -\dfrac{k}{GA}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}-\dfrac{(1-\beta)}{l^2}\dfrac{w_{i+1}-2w_i+w_{i-1}}{l^2}-\dfrac{k}{GA}\dfrac{1-\alpha}{l}\dfrac{w_{i+1}-w_{i-1}}{2l} \qquad (i=j+1) \\ +\dfrac{k}{EI}w_i+\dfrac{k(1-\beta)}{GAl^2}w_i=\dfrac{1}{EI}q_i+\dfrac{(1-\beta)}{GAl^2}q_i-\dfrac{1-\alpha}{GAl}\dfrac{q_{i+1}-q_{i-1}}{2l}-\dfrac{1}{GA}\dfrac{q_{i+1}-2q_i+q_{i-1}}{l^2}. \end{cases}$$

(B.2)

No virtual pressure, $e_1(x)$ or $e_2(x)$, was observed at the two ends of the shield tunnel. The four virtual nodes ($w_{-2}$, $w_{-1}$, $w_{n+1}$, and $w_{n+2}$) at each end of the tunnel were obtained because the boundary condition ensured that both the bending moments and shear forces were zero.

$$\begin{aligned} w_{-1}&=(2+\frac{kl^2}{GA})w_0-w_1+\frac{l^2}{GA}q_0, \\ w_{-2}&=(2+\frac{kl^2}{GA})w_0-2(2+\frac{kl^2}{GA})w_1+w_2+(2+\frac{kl^2}{GA})\frac{l^2}{GA}q_0+\frac{l^2}{GA}(q_1-q_{-1}), \\ w_{n+1}&=(2+\frac{kl^2}{GA})w_n-w_{n-1}+\frac{l^2}{GA}q_n, \\ w_{n+2}&=(2+\frac{kl^2}{GA})w_n-2(2+\frac{kl^2}{GA})w_{n-1}+w_{n-2}+(2+\frac{kl^2}{GA})\frac{l^2}{GA}q_n+\frac{l^2}{GA}(q_{n-1}-q_{n+1}). \end{aligned} \qquad \text{(B.3)}$$

The matrices and vectors in Eq. (20) are presented below.

$$[M_1] = \frac{1}{l^4}\begin{bmatrix} A_1 & A_2 & 2 & & & & & & \\ A_3 & 5 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 5 & A_3 \\ & & & & & & 2 & A_2 & A_1 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.4)}$$

$$[M_2] = \frac{k}{GAl^2}\begin{bmatrix} B & & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & & B \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.5)}$$

$$[M_3] = \frac{k}{EI}\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.6)}$$

$$[E_{11}] = \frac{(1-\alpha)}{2l^4}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{11}]_{(3\times5)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.7)}$$

$$[E_{12}] = \frac{(1-\alpha)k}{2GAl^2}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{12}]_{(3\times3)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.8)}$$

$$[E_{21}] = \frac{(1-\beta)}{l^4}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{21}]_{(3\times3)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.9)}$$

$$[E_{22}] = \frac{k(1-\beta)}{GAl^2}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{22}]_{(3\times3)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.10)}$$

$$[P_1] = \frac{1}{EI}\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.11)}$$

$$[P_2] = \frac{1}{GAl^2}\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.12)}$$

$$[QE_1] = \frac{(1-\alpha)}{2GAl^2}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{13}]_{(3\times 3)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.13)}$$

$$[QE_2] = \frac{(1-\beta)}{GAl^2}\begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & [D_{22}]_{(3\times 3)} & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad \text{(B.14)}$$

$$\{w\} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_{n-2} \\ w_{n-1} \\ w_n \end{bmatrix}_{1\times(n+1)}, \quad \text{(B.15)}$$

$$\{Q_1\} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_i \\ \vdots \\ q_{n-2} \\ q_{n-1} \\ q_n \end{bmatrix}_{1\times(n+1)} \quad , \text{(B.16)}$$

$$\{Q_2\} = \begin{bmatrix} q_1 - q_{-1} \\ q_2 - q_0 \\ q_3 - q_1 \\ \vdots \\ q_{i+1} - q_{i-1} \\ \vdots \\ q_{n-1} - q_{n-3} \\ q_n - q_{n-2} \\ q_{n+1} - q_{n-1} \end{bmatrix}_{1\times(n+1)} \quad , \text{(B.17)}$$

$$\{Q_3\} = \begin{bmatrix} q_1 - 2q_0 + q_{-1} \\ q_2 - 2q_1 + q_0 \\ q_3 - 2q_2 + q_1 \\ \vdots \\ q_{i+1} - 2q_i + q_{i-1} \\ \vdots \\ q_{n-1} - 2q_{n-2} + q_{n-3} \\ q_n - 2q_{n-1} + q_{n-2} \\ q_{n+1} - 2q_n + q_{n-1} \end{bmatrix}_{1\times(n+1)} \quad , \text{(B.18)}$$

$$\{Q_4\} = \begin{bmatrix} C_1 \\ C_2 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ C_3 \\ C_4 \end{bmatrix}_{1\times(n+1)} \quad , \text{(B.19)}$$

where

$$A_1 = \frac{kl^4 + 2(GA)^2}{(GA)^2}, A_2 = \frac{-2kl^2GA - 4(GA)^2}{(GA)^2}, A_3 = \frac{kl^2 - 2GA}{GA},$$

$$B = \frac{kl^2}{GA},$$

$$C_1 = \frac{GA(l^2q_0 - 2q_0 + q_1 - q_{-1}) + kl^2q_0}{l^2(GA)^2}, C_2 = \frac{q_0}{l^2GA},$$

$$C_4 = \frac{GA(l^2q_n - 2q_n + q_{n-1} - q_{n+1}) + kl^2q_n}{l^2(GA)^2}, C_3 = \frac{q_n}{l^2GA}, \quad \text{(B.20)}$$

$$[D_{11}] = \begin{bmatrix} -1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 & -1 \end{bmatrix}_{3\times5}, [D_{12}] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}_{3\times3}, [D_{13}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{3\times3}$$

$$[D_{21}] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}_{3\times3}, [D_{22}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3},$$

where [$D_{11}$], [$D_{12}$], [$D_{13}$], [$D_{21}$], and [$D_{22}$] are the stiffness matrices of the longitudinal joints. Considering an existing structure with multiple longitudinal joints, [$E_{11}$], [$E_{12}$], [$QE_1$], [$E_{21}$], [$E_{22}$], and [$QE_2$] will feature several [$D_{11}$], [$D_{12}$], [$D_{13}$], [$D_{21}$], and [$D_{22}$] at the positive diagonal lines, respectively.