Appendices for:

A numerical framework for underground structures in layered ground under inclined P-SV waves using stiffness matrix and domain reduction methods

Yusheng YANG[a,b], Haitao YU[c,d*], Yong YUAN[d], Dechun LU[e], Qiangbing HUANG[c]

[a] Department of Geotechnical Engineering, Tongji University, Shanghai 200092, China

[b] Shanghai Construction No.4(Group) Co., Ltd., Shanghai 201103, China

[c] Key Laboratory of Western China's Mineral Resources and Geological Engineering of Ministry of Education, Chang'an University, Xi'an 710054, China

[d] State Key Laboratory of Disaster Reduction in Civil Engineering, Tongji University, Shanghai 200092, China

[e] Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China

*Corresponding author. E-mail: yuhaitao@tongji.edu.cn

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Appendix A

The stiffness matrix of a homogeneous soil layer is presented here. With respect to deformation, plane strain is assumed in the *xz*-plane. Hence, the wave equations, in the absence of body forces, for a homogeneous elastic soil layer in the Cartesian coordinate system can be expressed as follows:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}=\rho_{\mathrm{s}}\frac{\partial^2 u_x}{\partial t^2},$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_z}{\partial z}=\rho_{\mathrm{s}}\frac{\partial^2 u_z}{\partial t^2} \quad , \text{(A1)}$$

where σ_x and σ_z denote the normal stresses in *x* and *z* directions, respectively; τ_{xz} denote the shear stress in *xz*-plane; u_x and u_z denote the displacements along *x* and *z* directions, respectively; ρ_{s} denotes the density of the soil layer; and *t* denotes time.

The constitutive relation of a homogeneous elastic soil layer is as follows:

$$\sigma_x=(\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_z}{\partial z}$$

$$\sigma_z=\lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z},$$

$$\tau_{xz}=\mu(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}) \quad , \text{(A2)}$$

where λ and μ denote the Lamé constants of the soil layer.

Substituting Eq. (A2) into Eq. (A1) leads to the following expression.

$$(\lambda+2\mu)\frac{\partial^2 u_x}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 u_z}{\partial x \partial z}+\mu\frac{\partial^2 u_x}{\partial z^2}=\rho_{\mathrm{s}}\frac{\partial u_x}{\partial t^2}$$

$$\mu\frac{\partial^2 u_z}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 u_x}{\partial x \partial z}+(\lambda+2\mu)\frac{\partial^2 u_z}{\partial z^2}=\rho_{\mathrm{s}}\frac{\partial u_z}{\partial t^2}. \text{ (A3)}$$

The double Fourier transform can be defined and inverted as follows:

$$\tilde{f}(k,\omega)=\boldsymbol{F}[f(x,t)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(x,t)\mathrm{e}^{-\mathrm{i}(\omega t-kx)}\mathrm{d}x\mathrm{d}t$$

$$f(x,t)=\boldsymbol{F}^{-1}[\tilde{f}(k,\omega)]=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}\,(k,\omega)\mathrm{e}^{\mathrm{i}(\omega t-kx)}\mathrm{d}k\mathrm{d}\omega, \text{ (A4)}$$

where $\boldsymbol{F}[\cdot]$and $\boldsymbol{F}^{-1}[\cdot]$denote the double Fourier transform and its inversion, respectively; *i* denotes the imaginary unit; ω denotes the circular frequency; and *k* denotes the wavenumber along the horizontal direction, which is as follows:

$$k=\frac{\omega}{V_{\mathrm{p}}}\cos\theta \quad \text{for P-wave,}$$

$$k=\frac{\omega}{V_{\mathrm{s}}}\cos\theta \quad \text{for SV-wave, (A5)}$$

where θ denotes the incident angle defined by the angle between the wave propagation direction and horizontal direction; and V_{p} and V_{s} denote the velocities of P-wave and SV-wave, respectively.

Applying the double Fourier transform to Eq. (A3) yields

$$k^2(\lambda+2\mu)\tilde{u}_x+\mathrm{i}k(\lambda+\mu)\frac{\partial \tilde{u}_z}{\partial z}-\mu\frac{\partial^2 \tilde{u}_x}{\partial z^2}-\rho_{\mathrm{s}}\omega^2\tilde{u}_x=0$$

$$k^2\mu\tilde{u}_z+\mathrm{i}k(\lambda+\mu)\frac{\partial \tilde{u}_x}{\partial z}-(\lambda+2\mu)\frac{\partial^2 \tilde{u}_z}{\partial z^2}-\rho_{\mathrm{s}}\omega^2\tilde{u}_z=0 \quad , \text{(A6)}$$

in which tilde "~" denotes the parallel variables in the frequency–wavenumber domain.

We can define the vector

$$\tilde{\boldsymbol{U}} = [\tilde{u}_x, -\mathrm{i}\tilde{u}_z]^{\mathrm{T}} \ , \text{(A7)}$$

in which the superscript "T" denotes the transposed matrix or vector. Thus, Eq. (A6) can be rewritten in matrix form as follows:

$$k^2\boldsymbol{D}_{xx}\tilde{\boldsymbol{U}} + k\boldsymbol{B}_{xz}\frac{\partial\tilde{\boldsymbol{U}}}{\partial z} - \boldsymbol{D}_{zz}\frac{\partial^2\tilde{\boldsymbol{U}}}{\partial z^2} - \rho_s\omega^2\tilde{\boldsymbol{U}} = 0 \ , \text{(A8)}$$

where

$$\boldsymbol{D}_{xx} = \begin{bmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{bmatrix}$$

$$\boldsymbol{B}_{xz} = \begin{bmatrix} 0 & -(\lambda + \mu) \\ \lambda + \mu & 0 \end{bmatrix} \ \text{(A9)}$$

$$\boldsymbol{D}_{zz} = \begin{bmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{bmatrix}.$$

Assuming $\tilde{\boldsymbol{U}}(k, \omega, z) = \tilde{\boldsymbol{U}}_0(k, \omega)\mathrm{e}^{nz}$ and substituting it into Eq. (A8) leads to the following expression.

$$(k^2\boldsymbol{D}_{xx} - n^2\boldsymbol{D}_{zz} + nk\boldsymbol{B}_{xz} - \rho_s\omega^2\boldsymbol{I})\tilde{\boldsymbol{U}}_0 = 0, \text{(A10)}$$

where $\boldsymbol{I}$ denotes the unit matrix. Eq. (A10) is the governing equation of the problem in the frequency–wavenumber domain, which is an eigenvalue problem with an eigenvalue of n and the corresponding eigenvector of $\tilde{\boldsymbol{U}}_0$.

By setting the coefficient determinant of Eq. (A10) as zero, we obtain

$$n = \pm kp, \pm ks \ , \text{(A11)}$$

and their corresponding eigenvectors can be obtained as follows:

$$\tilde{\boldsymbol{U}}_0 = [1, \pm p]^{\mathrm{T}}, [\pm s, 1]^{\mathrm{T}}, \text{(A12)}$$

where

$$p = \sqrt{1 - \left(\frac{\omega}{kV_{\mathrm{p}}}\right)^2}$$

$$s = \sqrt{1 - \left(\frac{\omega}{kV_{\mathrm{s}}}\right)^2}. \text{(A13)}$$

Therefore, the displacement responses can be obtained as follows:

$$\tilde{\boldsymbol{U}} = \boldsymbol{R}_1\boldsymbol{E}_z^{-1}\boldsymbol{A} + \boldsymbol{R}_2\boldsymbol{E}_z\boldsymbol{B}, \text{(A14)}$$

where

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & -s \\ -p & 1 \end{bmatrix}$$

$$\boldsymbol{R}_2 = \begin{bmatrix} 1 & s \\ p & 1 \end{bmatrix}$$

$$\boldsymbol{E}_z = \begin{bmatrix} \mathrm{e}^{kpz} & 0 \\ 0 & \mathrm{e}^{ksz} \end{bmatrix}$$

$$\boldsymbol{A} = [A_{\mathrm{p}}, A_{\mathrm{s}}]^{\mathrm{T}}$$

$$\boldsymbol{B} = [B_{\mathrm{p}}, B_{\mathrm{s}}]^{\mathrm{T}}, \text{(A15)}$$

where A_{p}, B_{p}, A_{s}, and B_{s} denote unknown constants pertinent to the upward and downward wave amplitudes, and superscript "-1" denotes the inverse matrix.

By applying the double Fourier transform to Eq. (A2), the following expression can be obtained.

$$\tilde{\sigma}_z = -\mathrm{i}k\lambda\tilde{u}_x + (\lambda + 2\mu)\frac{\partial\tilde{u}_z}{\partial z}$$

$$\tilde{\tau}_{xz} = \mu \frac{\partial \tilde{u}_x}{\partial z} - \mathrm{i}k\mu\tilde{u}_z. \text{ (A16)}$$

Analogously, by defining $\tilde{\boldsymbol{S}} = [\tilde{\tau}_{xz}, -\mathrm{i}\tilde{\sigma}_z]^{\mathrm{T}}$ and substituting it into Eq. (A16), we obtain the following

$$\tilde{\boldsymbol{S}} = k\mu[-\boldsymbol{Q}_1\boldsymbol{E}_z^{-1}\boldsymbol{A} + \boldsymbol{Q}_2\boldsymbol{E}_z\boldsymbol{B}], \text{ (A17)}$$

where

$$\boldsymbol{Q}_1 = \begin{bmatrix} 2p & -(1+s^2) \\ -(1+s^2) & 2s \end{bmatrix}$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 2p & 1+s^2 \\ 1+s^2 & 2s \end{bmatrix}. \text{ (A18)}$$

A horizontal homogeneous soil layer with thickness h, as shown in Fig. A1, is considered. Assigning suffixes 1 and 2 to the upper and lower surfaces yields:

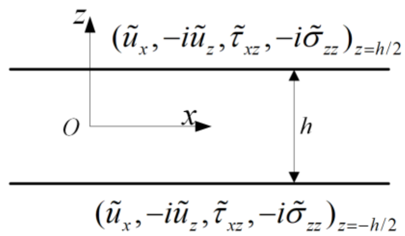

Fig. A1 Displacements and stresses of a single layer.

$$\begin{bmatrix} \tilde{\boldsymbol{U}}_1 \\ \tilde{\boldsymbol{U}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{E}_{\mathrm{h}/2}^{-1} & \boldsymbol{R}_2\boldsymbol{E}_{\mathrm{h}/2} \\ \boldsymbol{R}_1\boldsymbol{E}_{-\mathrm{h}/2}^{-1} & \boldsymbol{R}_2\boldsymbol{E}_{-\mathrm{h}/2} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix}, \text{ (A19)}$$

$$\begin{bmatrix} \tilde{\boldsymbol{S}}_1 \\ -\tilde{\boldsymbol{S}}_2 \end{bmatrix} = k\mu \begin{bmatrix} -\boldsymbol{Q}_1\boldsymbol{E}_{\mathrm{h}/2}^{-1} & \boldsymbol{Q}_2\boldsymbol{E}_{\mathrm{h}/2} \\ \boldsymbol{Q}_1\boldsymbol{E}_{-\mathrm{h}/2}^{-1} & -\boldsymbol{Q}_2\boldsymbol{E}_{-\mathrm{h}/2} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix}, \text{ (A20)}$$

$$\begin{bmatrix} \tilde{\boldsymbol{S}}_1 \\ -\tilde{\boldsymbol{S}}_2 \end{bmatrix} = \boldsymbol{K} \begin{bmatrix} \tilde{\boldsymbol{U}}_1 \\ \tilde{\boldsymbol{U}}_2 \end{bmatrix}, \text{ (A21)}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix} = k\mu \begin{bmatrix} -\boldsymbol{Q}_1\boldsymbol{E}_{\mathrm{h}/2}^{-1} & \boldsymbol{Q}_2\boldsymbol{E}_{\mathrm{h}/2} \\ \boldsymbol{Q}_1\boldsymbol{E}_{-\mathrm{h}/2}^{-1} & -\boldsymbol{Q}_2\boldsymbol{E}_{-\mathrm{h}/2} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{E}_{\mathrm{h}/2}^{-1} & \boldsymbol{R}_2\boldsymbol{E}_{\mathrm{h}/2} \\ \boldsymbol{R}_1\boldsymbol{E}_{-\mathrm{h}/2}^{-1} & \boldsymbol{R}_2\boldsymbol{E}_{-\mathrm{h}/2} \end{bmatrix}^{-1}, \text{ (A22)}$$

where $\boldsymbol{K}$ denotes the dynamic stiffness matrix of a single horizontal homogeneous elastic layer, which is a 4×4 symmetric matrix dependent on the wavenumber, frequency, thickness, and material parameters of the soil layer.

For the underlying half-space, the displacements and stresses at negative infinity are zero. Thus, its stiffness matrix can be obtained as follows:

$$\boldsymbol{K}_{\mathrm{half}} = \frac{k\mu(1-s^2)}{1-ps} \begin{bmatrix} p & -1 \\ -1 & s \end{bmatrix} + 2k\mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \text{ (A23)}$$

Appendix B

The equivalent nodal forces in the DRM are presented in this section. As shown in Fig. 3, the discretized finite element equations, neglecting the damping terms, of the interior domain Ω^- and exterior domain Ω^+ are as follows:

$$\begin{bmatrix} \boldsymbol{M}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{ib}}^{\Omega^-} \\ \boldsymbol{M}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{bb}}^{\Omega^-} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\mathrm{i}} \\ \ddot{\boldsymbol{u}}_{\mathrm{b}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{ib}}^{\Omega^-} \\ \boldsymbol{K}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{bb}}^{\Omega^-} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{i}} \\ \boldsymbol{u}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{F}_{\mathrm{b}} \end{bmatrix} \text{ in } \Omega^-, \text{ (B1)}$$

$$\begin{bmatrix} \boldsymbol{M}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \\ \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\mathrm{b}} \\ \ddot{\boldsymbol{u}}_{\mathrm{e}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \\ \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{b}} \\ \boldsymbol{u}_{\mathrm{e}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{F}_{\mathrm{b}} \\ \boldsymbol{F}_{\mathrm{e}} \end{bmatrix} \text{ in } \Omega^+, \text{ (B2)}$$

where $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{F}$ denote the mass matrix, stiffness matrix, and load vector, respectively; $\ddot{\boldsymbol{u}}$ and $\boldsymbol{u}$ denote the acceleration and displacement vectors, respectively; and subscripts *i*, *b*, and *e* refer to the nodes in the interior domain, boundary Γ, and exterior domain, respectively. Superscripts Ω^- and Ω^+ denote matrices of elements in either the interior or exterior domain.

Assuming that the ground in the exterior domain is elastic, the displacement in the exterior domain can therefore be expressed by the superposition of displacements of the free field and the residual field as follows:

$$\boldsymbol{u}_{\mathrm{e}} = \boldsymbol{u}_{\mathrm{e}}^{\mathrm{f}} + \boldsymbol{w}_{\mathrm{e}}, \text{ (B3)}$$

where residual field $\boldsymbol{w}_{\mathrm{e}}$ denotes the difference in the response in the exterior domain with respect to that of the free field $\boldsymbol{u}_{\mathrm{e}}^{\mathrm{f}}$, which can be due to scattered waves or soil nonlinearity in the interior domain. It should be noted that heterogeneous ground can be incorporated into the stiffness matrix method as stated above. Therefore, the residual field $\boldsymbol{w}_{\mathrm{e}}$ is significantly reduced, thereby decreasing the outgoing waves. Substituting Eq. (B3) into Eq. (B2) and adding Eq. (B1) yields:

$$\begin{bmatrix} \boldsymbol{M}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{ib}}^{\Omega^-} & 0 \\ \boldsymbol{M}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{bb}}^{\Omega^-} + \boldsymbol{M}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \\ 0 & \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\mathrm{i}} \\ \ddot{\boldsymbol{u}}_{\mathrm{b}} \\ \ddot{\boldsymbol{w}}_{\mathrm{e}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{ib}}^{\Omega^-} & 0 \\ \boldsymbol{K}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{bb}}^{\Omega^-} + \boldsymbol{K}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \\ 0 & \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{i}} \\ \boldsymbol{u}_{\mathrm{b}} \\ \boldsymbol{w}_{\mathrm{e}} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -\boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{e}}{}^{\mathrm{f}} - \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \boldsymbol{u}_{\mathrm{e}}{}^{\mathrm{f}} \\ \boldsymbol{F}_{\mathrm{e}} - \boldsymbol{M}_{\mathrm{ee}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{e}}{}^{\mathrm{f}} - \boldsymbol{K}_{\mathrm{ee}}^{\Omega^+} \boldsymbol{u}_{\mathrm{e}}{}^{\mathrm{f}} \end{bmatrix}. \text{ (B4)}$$

The following observation from Eq. (B2) can be considered.

$$\boldsymbol{F}_{\mathrm{e}} = \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{b}}{}^{\mathrm{f}} + \boldsymbol{M}_{\mathrm{ee}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{e}}{}^{\mathrm{f}} + \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} \boldsymbol{u}_{\mathrm{b}}{}^{\mathrm{f}} + \boldsymbol{K}_{\mathrm{ee}}^{\Omega^+} \boldsymbol{u}_{\mathrm{e}}{}^{\mathrm{f}}. \text{ (B5)}$$

Substituting Eq. (B5) into Eq. (B4) yields:

$$\begin{bmatrix} \boldsymbol{M}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{ib}}^{\Omega^-} & 0 \\ \boldsymbol{M}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{M}_{\mathrm{bb}}^{\Omega^-} + \boldsymbol{M}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \\ 0 & \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{M}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_{\mathrm{i}} \\ \ddot{\boldsymbol{u}}_{\mathrm{b}} \\ \ddot{\boldsymbol{w}}_{\mathrm{e}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{\mathrm{ii}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{ib}}^{\Omega^-} & 0 \\ \boldsymbol{K}_{\mathrm{bi}}^{\Omega^-} & \boldsymbol{K}_{\mathrm{bb}}^{\Omega^-} + \boldsymbol{K}_{\mathrm{bb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \\ 0 & \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} & \boldsymbol{K}_{\mathrm{ee}}^{\Omega^+} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{i}} \\ \boldsymbol{u}_{\mathrm{b}} \\ \boldsymbol{w}_{\mathrm{e}} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -\boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{e}}{}^{\mathrm{f}} - \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \boldsymbol{u}_{\mathrm{e}}{}^{\mathrm{f}} \\ \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{b}}{}^{\mathrm{f}} + \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} \boldsymbol{u}_{\mathrm{b}}{}^{\mathrm{f}} \end{bmatrix}. \text{ (B6)}$$

The left side of Eq. (B6) is identical to that of the sum of Eq. (B1) and Eq. (B2). Hence, the seismic excitation in the exterior domain, regardless of its location, is replaced by equivalent nodal forces on the right side. The equivalent nodal forces $P_{\mathrm{b}}{}^{\mathrm{eff}}$ are imposed on the nodes located on boundaries Γ and Γ_{e} and those enclosed by two boundaries

$$\begin{bmatrix} \boldsymbol{P}_{\mathrm{b}}{}^{\mathrm{eff}} \\ \boldsymbol{P}_{\mathrm{e}}{}^{\mathrm{eff}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{M}_{\mathrm{be}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{e}}{}^{\mathrm{f}} - \boldsymbol{K}_{\mathrm{be}}^{\Omega^+} \boldsymbol{u}_{\mathrm{e}}{}^{\mathrm{f}} \\ \boldsymbol{M}_{\mathrm{eb}}^{\Omega^+} \ddot{\boldsymbol{u}}_{\mathrm{b}}{}^{\mathrm{f}} + \boldsymbol{K}_{\mathrm{eb}}^{\Omega^+} \boldsymbol{u}_{\mathrm{b}}{}^{\mathrm{f}} \end{bmatrix}. \text{ (B7)}$$

It can be observed that the equivalent nodal forces are dependent on the free-field displacements and accelerations, location, and material parameters of the elements enclosed by the two boundaries Γ and Γ_{e}. In the numerical implementation, only a single-element thick layer is considered between the two boundaries Γ and Γ_{e} for simplicity. This implies that there are no other enclosed nodes, and equivalent nodal forces are imposed only on the two boundaries.